Classical and Quantum aspects of Field theory compactification

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2025 年 6 月 6 日

I will use $diag(-1,1,1,1)$ in this short article.

目录

1 Kaluza-Klein theory

1.1 Kaluza-Klein metric

We use this renowned theory as our beginning, the Kaluza-Klein theory is the first theory that uses the concept of compactification. It assumes its spacetime as the flat Minkowski-like spacetime direct times a compact space, which gives people more freedom to construct their theory in a higher dimension, and later compactifies to the regular dimension.

What Kaluza proposed in 1919 is a theory that has a different dimension with all physics theory at that time, it assumed a spacetime which has the global topology as $M^4 \times S^1$ and the entire 5d metric is

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix} \tag{1}$$

and its inverse is

$$\tilde{g}^{\mu\nu} = \begin{pmatrix} g_{\mu\nu} & -A^\nu \\ -A^\mu & \phi^{-2} + A_\mu A^\mu \end{pmatrix} \tag{2}$$

and its determinant is $\tilde{g} = \phi^2 g$. A important assumption in all theory with extra dimension or compactified dimension is the higher dimension field should be free from compactified dimension too keep the background stable,

$$\partial_5 \tilde{g}_{\mu\nu} = 0 \tag{3}$$

It's not hard to see that this metric choice has a $U(1)$ symmetry inside the compactified dimension, explicitly, when we consider the transformation

$$(x^\mu)' \to x^\mu \tag{4}$$

$$(x^5)' \to x^5 - \chi(x^\mu) \tag{5}$$

and when we examine how the metric transforms

$$\tilde{g}'_{\mu\nu} = \frac{\partial \tilde{x}^\alpha}{\partial \tilde{x}'^\mu} \frac{\partial \tilde{x}^\beta}{\partial \tilde{x}'^\nu} \tilde{g}_{\alpha\beta} \tag{6}$$

we get

$$\phi \to \phi \quad A_\mu \to A_\mu + \partial_\mu \chi \quad g_{\mu\nu} \to g_{\mu\nu} \tag{7}$$

where $g_{\mu\nu}$ is the 4d metric.This is a sign that when we treat the Kaluza-Klein theory as a 4d theory with extra dimension, we will gain a extra gauge theory as well.We can also easily calculate its connection using the standard method in general relativity,

$$\begin{aligned} \tilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \phi^2 g^{\alpha\beta}(F_{\mu\beta}A_\nu + A_\mu F_{\nu\beta} - A_\mu A_\nu \partial_\beta ln\phi^2) \\ \tilde{\Gamma}^\alpha_{\mu 5} = \frac{1}{2} g^{\alpha\beta}\phi^2(F_{\mu\beta} - A_\mu \partial_\beta ln\phi^2) \\ \tilde{\Gamma}^\alpha_{55} = -\frac{1}{2} g^{\alpha\beta}\partial_\beta \phi^2 \end{aligned} \tag{8}$$

1.2 Kaluza-Klein action

The Kaluza-Klein action is merely a 5d version of 4d Einstein-Hilbert action,

$$\tilde{S}_{KK} = \int \frac{1}{2\tilde{\kappa}} \tilde{R} \sqrt{\tilde{g}} d^5 x \tag{9}$$

and we can write the 5d scalar curvature in terms of 4d scalar curvature

$$\tilde{R} = R - \frac{1}{4}\phi^2 F_{\mu\nu}F^{\mu\nu} - \frac{2}{\phi}\partial_\mu\partial^\mu\phi \tag{10}$$

$$\tilde{S}_{KK} = \int [\frac{1}{2\tilde{\kappa}}R - \frac{1}{8\tilde{\kappa}}\phi^2 F_{\mu\nu}F^{\mu\nu}]\phi\sqrt{g}d^5x - \int \frac{1}{\tilde{\kappa}}\partial_\mu\partial^\mu\phi\sqrt{g}d^5x \tag{11}$$

when the theory is compactified, the extra dimension is integrated and becomes a constant as the coefficient of the 4d action

$$\tilde{S}_{KK} = \int [\frac{C}{2\tilde{\kappa}}R - \frac{C}{8\tilde{\kappa}}\phi^2 F_{\mu\nu}F^{\mu\nu}]\phi\sqrt{g}d^4x - \int \frac{C}{\tilde{\kappa}}\partial_\mu\partial^\mu\phi\sqrt{g}d^4x \tag{12}$$

with proper redefinition of the vector field and coefficients

$$A_\mu \to \sqrt{\frac{2\tilde{\kappa}}{C}}A_\mu \quad \tilde{\kappa} = 8\pi GC \tag{13}$$

it's not surprised to find out that if we assume the scalar field merely changes(as a constant), the 4d action is Einstein-Hilbert action together with Maxwell action

$$\mathcal{L} = \frac{1}{16\pi G}R\sqrt{g} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{14}$$

1.3 natural charge quantization due to compactification

When we examine the 5d geodesics,

$$\frac{d\tilde{U}^\mu}{ds} + \Gamma^\mu_{\alpha\beta}\tilde{U}^\alpha\tilde{U}^\beta = \sqrt{8\pi G}\phi^2\tilde{U}^5 F^\mu_\alpha\tilde{U}^\alpha \tag{15}$$

and redefine

$$\frac{q}{m} = \sqrt{8\pi G}\tilde{U}^5 \tag{16}$$

we actually get a 5d form of Lorentz force, but when the extra dimension is a periodic circle, the extra momentum is naturally discrete

$$\frac{2\pi}{m\tilde{U}^5} = \frac{C}{n} \tag{17}$$

thus

$$q = \frac{2\pi n}{C}\sqrt{8\pi G} \tag{18}$$

makes the charge quantized

2 Beyond Kaluza Klein:how extra dimension contribute to observable physics

2.1 extra scalar particles and harmonic operators

Kaluza-Klein theory proposed the higher dimensional field do not depends with the extra dimension, but since the 5d Kaluza-Klein field is the metric field which is the background as well as the dynamical degree of freedoms, it's not the general case that all higher dimensional field should be fixed into such strict condition.

One general guess of how higher dimensional field are compactified to lower dimension and makes lower dimensional field free from compactified coordinates is by keep all dependence of compactified coordinates inside the compact space,we still use the circle to be the compactified space as a example.

$$S = \int d^4xdy\frac{1}{2}|\partial_M\Phi|^2 \tag{19}$$

and the fields dependence of the fifth dimension can be expanded

$$\Phi(x^\mu, y) = \sum_n \phi_n(x^\mu)e^{iny/R} \tag{20}$$

considering that

$$\int_0^{2\pi} e^{iny/R}e^{-imy/R} = 2\pi\delta_{nm} \tag{21}$$

we get the 4d effective theory

$$S_{eff} = \int d^4x \sum_n ((\partial_\mu\phi_n)^2 + \frac{n^2}{R^2}\phi_n^2) \tag{22}$$

A question occurs when we generalized our compactified space beyong a circle, generally, a higher dimensional compact differential manifold, for example we can do

$$\Phi(x^\mu, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{lm}(x^\mu) Y_{lm}(\theta, \phi) \tag{23}$$

where Y_{lm} is the spherical harmonics, and this gives a family of particles

$$S_{eff} = \int d^4 \sum_{l,m} [(\partial_\mu \phi_{lm})^2 + \frac{l(l+1)}{R^2} \phi_{l,m}^2] \tag{24}$$

things becomes clear when we see the appearance of spherical harmonics, as the solution of Laplace-Beltrami equation on the sphere, we can easily assume that basis on each manifold is the solution of Laplace-Beltrami equation on each manifold

$$\Delta f = \lambda f \tag{25}$$

the Laplace-Beltrami operator on any smooth Riemann manifold can be represent as

$$\Delta_g f = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j f) \tag{26}$$

the higher dimensional field can be there expanded

$$\Phi(x^\mu, y^i) = \sum_k \phi_k(x^\mu) f_k(y^i) \tag{27}$$

where f_k are the solutions which has eigenvalue λ_k, and the compactification gives the whole family of particles that has mass

$$m_k^2 = -\lambda_k \tag{28}$$

2.2 how about vectors and spinors

We have go over the case for scalars fields, but what for vector and spinor fields?Recall that the scalar field has to satisfy a classical Klein-Gordon equation:

$$(\Box + m^2)\phi(x^\mu) = 0 \tag{29}$$

and this shows why the basis of expanding the higher dimension scalar field requires the participation of solution of Laplace-Beltrami equation.

2.2.1 spinors

For a higher dimensional dirac action:

$$S_f = \int d^D x \sqrt{-G}\bar{\Psi}\Gamma^M D_M \Psi \tag{30}$$

where D_M is the covariant derivative with the participation of spin connection

$$D_M = \partial_M + \omega_M \tag{31}$$

then we can perform our expansion by basis functions

$$\Psi(x^\mu, y^i) = \sum_k \psi^{(k)}(x^\mu) \otimes f_k(y^i) \tag{32}$$

where f_k is the eigenfunction of Dirac equation

$$i\not{D} f_k(y^i) = m_k f_k(y^i) \tag{33}$$

and lower dimensional effective theory enjoys a spectra of mass m_k spinor field.

When does a lower dimension spinor is Majorana/Weyl/Majorana-Weyl? Recall the Majorana condition:

$$\Psi = \Psi^c = C\Psi^\dagger \tag{34}$$

and Weyl decomposition

$$P_\pm = \frac{1 \pm \Gamma_{D+1}}{2} \tag{35}$$

where

$$\Gamma_{D+1} \propto \prod_{M=1}^{D} \Gamma_M \tag{36}$$

due to the representation property of Clifford algebra (specifically the property of Bott perodicity), we know that only even dimensions admits a Weyl decomposition and only $D = 0, 1, 2, 3, 4 mod 8$ admits a Majorana spinor. Specially, only $D = 2 mod 8$ admits a Majorana-Weyl spinor.

Thus, only if the original spinor is Majorana and

$$C_{total} = C_{internal} \otimes C_{spacetime} \tag{37}$$

and $C_{spacetime}$ is in a dimension where Majorana condition is well defined, the compactified spinor can be Majorana.

The case for Weyl spinor compactification has some rather complicated issues and we leave to the next subsection.

2.2.2 vectors

For gauge vectors,we do the decomposition

$$A_M(x^\mu, y^i) = (A_\mu(x^\mu, y^i), A_i(x^\mu, y^i)) \tag{38}$$

where the latter terms are those components on the compactified space,we decompose the vector part still using the harmonics

$$A_\mu(x^\mu, y^i) = \sum_k A_\mu^{(k)}(x^\mu) f_k(y^i) \tag{39}$$

still, these vector fields has

$$m_k^2 = \lambda_k \tag{40}$$

and only the zero modes persists the gauge symmetry.For the scalar part, we some times choose the basis as the solution of

$$\nabla^a v_a^{(k)}(y^i) = 0 \quad (\Box + m_k^2) v_a^k = 0 \tag{41}$$

and

$$A_a(x^\mu, y^i) = \sum_k \phi^{(n)}(x^\mu) v_a^{(k)}(y^i) \tag{42}$$

where the scalar mass is zero if there exists Killing vectors.

Actually those zero mass scalar fields are the generalization of Wilson integral on a circle, which still describes how the gauge field is curved on the extra dimension which cannot be eliminated by gauge transformations.

2.3 We can't hear the shape of a drum

If we already know all the observable spectra of the effective theory,can we exactly fix the full theory before compactification? Luckily, this question is studied by some of the greatest mathematicians of the last century.

2.3.1 0-modes and cohomology

From a general analysis of Lapalace-Beltrami operator, we are able to know every of its eigenvalues associate to a finite dimensional eigenvector space which often has degeneracy.The degeneracy of the zero mode can be easily counted by Hodge's theorem of decomposition

$$H^k_{dR}(M) \simeq \mathcal{H}^k(M) \tag{43}$$

which states the k-th order harmonic forms has a isomorphic space to k-th order de Rham cohomology group.This means the degeneracy of zero mode is simply b^0, which is determined from the connectness.Since most of our cases is connected, it's a good news for us that a higher dimensional gauge field only generates one lower dimensional gauge field, as the vector part is expanded by harmonics:

$$A_\mu(x^\mu, y^i) = \sum_k A_\mu^{(k)}(x^\mu) f_k(y^i) \tag{44}$$

2.3.2 Spinors and Index theorem and other invariants

We should note that not all manifolds is compatible with spinors.Since the spin group is a 2-times cover of the special orthogonal group, there's some conditions when the vector bundles can be lift to a spin bundle, and one summarized condition is that

$$w_2(M) = 0 \tag{45}$$

which is the trivialness of the second Stiefel-Whitney class.Only on these manifolds we can discuss the definition and solution of Dirac equation.

One spectacular result of Dirac operator is the Index theorem, which defines the index of Dirac operator as

$$ind(D) = \begin{cases} dimker(D^+) - dimcoker(D^+), dim(M) = 2n \\ 0, dim(M) = 2n + 1 \end{cases}$$

where on even dimensional Spin manifolds the spin bundle can be Weyl decomposed, and

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \tag{46}$$

the index theorem states that

$$\int_M \hat{A}(TM) \tag{47}$$

where

$$\hat{A}(TM) = \prod_{j=1}^{n/2} \frac{x_j/2}{sinh(x_j/2)} \tag{48}$$

where x_j is the Chern root from the Chern class.This strictly constrains the space of zero mass fermions.

We also have some estimation of non zero mass (but the lightest Dirac fermion), one estimation from Lichnerowicz is

$$m_{min} \geq \frac{R}{4} \tag{49}$$

where associate the lightest mass to the manifold's scalar curvature.

2.3.3 We can't hear the shape of a drum

A natural question occurs after we analyzed the spectra of the effective theory and the compact space: Can we figure out what exactly is the compact space if we have all the spectra information? This coincident to a famous article from Kac discussing the eigenvalues and the geometric and topological information of where the Laplace equation is solved, sadly proved by Milnor and other mathematicians, there are some examples when two different spaces can give identical eigenvalues.

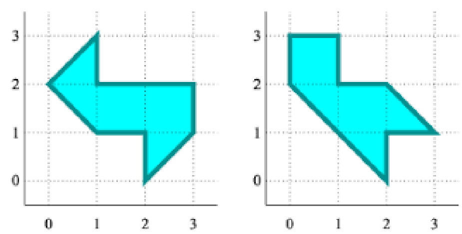

图 1: these two spaces have the same eigenvalues

However there are lot of information of the compact space can be induced from the eigenvalue, like the Volume can be estimated from the Weyl's theorem

$$N(\lambda) = \#\{\lambda' < \lambda\} \sim \frac{Vol(M)}{(2\pi)^n} Vol(S^n)\lambda^{n/2} \tag{50}$$

and also the heat kernel's expansion associates to various of topological invariants

$$\sum_k e^{-\lambda_k t} \sim \frac{1}{(4\pi t)^{n/2}} \sum_{j=0}^{\infty} a_j t^j \tag{51}$$

all these results means although we cannot exactly remodels the full theory in higher dimension, we can make our best to guess and know as much as its information from the generated particles from compactification.

3 Interaction in compactification

We still use the eigenfunction of the Laplace-Beltrami equation as the basis of expansion

$$\Delta f_n(y^i) = -\lambda_n f_n(y^i) \tag{52}$$

and a general Phi4 theory is

$$S = \int d^4x \int_M d^dy\sqrt{g}[\frac{1}{2}\partial_M\phi\partial^M\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4] \tag{53}$$

when we regard the minus modes to be the conjugation, we find out the mass of effective fields to be

$$m_n^2 = m^2 + \lambda_n \tag{54}$$

when we compactify the interaction term

$$\begin{aligned} S_{int} &= \frac{\lambda}{4!}\int d^4x \int_M d^dy\sqrt{g}\phi^4 \\ &= \frac{\lambda}{4!}\int d^4x \sum_{n_1,n_2,n_3,n_4} \phi_{n_1}\phi_{n_2}\phi_{n_3}\phi_{n_4}\int_M d^dy\sqrt{g}f_{n_1}f_{n_2}f_{n_3}f_{n_4} \end{aligned} \tag{55}$$

this gives

$$S_{int} = \frac{\lambda}{4!}\int d^4x \sum_{n_1,n_2,n_3,n_4} C_{n_1n_2n_3n_4}\phi_{n_1}\phi_{n_2}\phi_{n_3}\phi_{n_4} \tag{56}$$

where

$$C_{n_1n_2n_3n_4} = \int_M d^dy\sqrt{g}f_{n_1}f_{n_2}f_{n_3}f_{n_4} \tag{57}$$

this produce selection rule on some basis where more symmetries is admitted, for on S^1,only

$$n_1 + n_2 + n_3 + n_4 = 0 \tag{58}$$

is allowed,similarly when we consider Yukawa theory or QED theory compactified on a arbitrary compact space, we should calculate(we use Yukawa as a example)

$$C_{nmk} = \int_M d^dd\sqrt{g}g_n^\dagger g_m f_k \tag{59}$$

where g_n is the solution of Dirac equation

4 Compactification after quantization

4.1 General description

We know that, generally, the measure of path integral changes like

$$\mathcal{D}\phi = \mathcal{D}\chi |J(\frac{\delta\phi}{\delta\chi})| exp(i \int \mathcal{A}[\chi]) \tag{60}$$

for we change our field description from $\{\phi\}$ to $\{\chi\}$, specifically in the case of compactification,

$$\Phi(x, y) = \sum_k \phi_k(x) f_k(y) \tag{61}$$

the Jacobian

$$J_k(x, y) = \frac{\delta\Phi(x, y)}{\delta\phi_k(x')} = f_k(y)\delta^{(d)}(x - x') \tag{62}$$

so for most our cases, when normalization is well performed, we have

$$\mathcal{D}\Phi = \prod_k \mathcal{D}\phi_k \tag{63}$$

but in some cases, there is a quantum anomaly comes from compactifying the higher dimensional field.

4.2 Anomaly by Compactification

However, it's very simple to construct a theory which has a quantum anomaly after compactification.We consider a massless 6d QED theory

$$S_{6D} = \int d^4xdy(-\frac{1}{4}F_{MN}F^{MN} + i\bar{\Psi}\gamma^M D_M\Psi) \tag{64}$$

We know that normally there is a chiral anomaly in 4d QED theory, so we have to find out how to make this theory anomaly in 4d.

We choose an orbifold $T^2/\mathbb{Z}_2$ to define its compact space, this space makes $y = (0, 0)$, $y = (0, \pi R_2)$,$y = (\pi R_1, 0)$， $y = (\pi R_1, \pi R_2)$ four singular point of the space. We can specifically fix how the Fermion behaves on the orbifold

$$\Psi(x,-y) = \eta\gamma_5\Psi(x,y) \tag{65}$$

and for unitary considerations, we have

$$\eta = \pm 1 \tag{66}$$

We can apply different boundary conditions to left and right hand side Weyl fermions

$$\Psi(x,-y)_L = \Psi(x,y)_R \tag{67}$$

and

$$\Psi(x,-y)_R = \Psi(x,y)_R \tag{68}$$

this means when compactifying the theory and by only considering it's zero mode, only right hand fermion will survive in the theory

$$S_{4D} = \int d^4x dy(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}_R\gamma^\mu D_\mu \Psi_R) \tag{69}$$

and this gives a chiral anomaly.

5 a short example of Flux compactification

One interesting aspect on modern string compactification is the flux compactification, and I will here present a short example on this method.

Consider a 5d Maxwell-Chern-Simons theory which involves a Chern-Simons interacting term

$$S_{5d} = \int d^5x(-\frac{1}{4}F_{MN}F^{MN} + \frac{\kappa}{2}\epsilon^{MNPQR}A_M\partial_N A_P\partial_Q A_R) \tag{70}$$

and we compactify the theory on a circle, for vector fields, we often do the decomposition

$$A_\mu(x^\mu, y) = A_\mu(x^\mu) \quad A_y(x^\mu, y) = \phi(x^\mu) \tag{71}$$

and here we only consider the Kaluza-Klein zero mode, higher modes are not considered. The idea of flux compactification is by introducing a nontrivial source on the compactified dimension

$$\int_{S^1} dy \partial_y A_y = 2\pi n \tag{72}$$

where

$$A_y = \frac{n}{R} + \phi(x^\mu) \tag{73}$$

and this not only gives a scalar field ϕ, but also introducing a new parameter which can control the interaction

$$S_{4D} = \int d^4x(-\frac{\pi R}{2} F_{\mu\nu}F^{\mu\nu} - \pi R(\partial_\mu \phi)^2 + \frac{\kappa n}{R}\epsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu A_\rho \partial_\sigma \phi) \tag{74}$$

in actual physics, vector fields are some times fluxed by a generalization version

$$\int_{\Sigma_p} F_p = n \tag{75}$$

where the element Σ_p is often chosen as an element in the Homology group as a generalization of non trivial circle.

6 How far can Kaluza-Klein goes

6.1 non abelian gauge theories from Kaluza-Klein theory

In order to construct non abelian gauge theories from the standard Kaluza-Klein technique, we need to make the compact manifold has some nontrivial properties, one important property, since Kaluza-Klein theory always expect a unification with Einstein gravity theory, is the group of isometry on the compact space.

$$y \to y' : \tilde{g}'_{mn}(y') = \tilde{g}_{mn}(y) \tag{76}$$

an isometry is a coordinate that leaves the form of the metric invariant, and when the general infinitesimal isometry is

$$e^{i\epsilon^a t_a} : y^n \to y^n + \epsilon^a \xi_a^n(y) \tag{77}$$

where ξ_a^n are the Killing vectors,they obey the algebra

$$\xi_b^m \partial_m \xi_b^n - \xi_c^m \partial_m \xi_b^n = -C_{abc}\xi_a^n \tag{78}$$

where

$$[t_a, t_b] = iC_{abc}t_c \tag{79}$$

It is obvious since we want a non abelian gauge field, we may need a compact manifold which has dimensions more then 1. And the full metric is assumed as

$$\begin{pmatrix} g_\mu\nu - \tilde{g}_{mn}(y)B_\mu^m B_\nu^n & B_\mu^n \\ B_\nu^m & -\tilde{g}_{mn}(y) \end{pmatrix} \tag{80}$$

where

$$B_\mu^n = \xi_a^n A_\mu^a \tag{81}$$

and this gives A_μ^a a non abelian gauge symmetry

$$A_\mu^a \to A_\mu^a + \partial_\mu \epsilon^a(x) + C_{abc}\epsilon^b(x)A_\mu^c \tag{82}$$

we can re define the generators by

$$C_{abc} = gf_{abc} \quad t_a = gT_a \tag{83}$$

and this introduces the coupling constant and group generators $\{T_a\}$

6.2 Can the standard model be constructed from Kaluza-Klein

So the construction above shows that gauge group of lower dimensions can be viewed as a isometry group of compact space, if we want to directly get a standard model from Kaluza-Klein, we need to find a space that has $SU(3) \times SU(2) \times U(1)$.

One construction of spaces with an arbitrary isometry group G is by constructing G/H where H is the subgroup of G, from this we can estimate the dimension of the expected compact space, where the maximal subgroup of $SU(3) \times SU(2) \times U(1)$ is

$$H = SU(2) \times U(1) \times U(1) \tag{84}$$

and

$$dim(K) = dim(G/H) = dim(G) - dim(H) = 12 - 5 = 7 \tag{85}$$

this means the unified theory should at least be 11 dimensional to generate the standard model. Witten studied this question at 1981, and he denote the general coset space of this type as space M^{pqr}, when denoting the generators of $SU(3)$, $SU(2)$ and $U(1)$ by

$$\frac{1}{2}\lambda_a, \frac{1}{2}\sigma_\alpha, Y \tag{86}$$

and it is necessary to select two $U(1)$ factors commuting with the $SU(2)$ to be in H. In other words, it is necessary to select one combination of $\frac{1}{2}\lambda_8, \frac{1}{2}\sigma_3, Y$which does not occur in H, Witten denote this combination as

$$Z = \frac{1}{2}(\sqrt{3}p\lambda_8 + q\sigma_3 + 2rY) \tag{87}$$

with p, q, r are arbitrary integers to give a compact $U(1)$, then the two combinations which lie in H may be taken to be ther orthogonal combinations

$$Z' = \frac{1}{2}(2\sqrt{3}pr\lambda_8 + 2qr\sigma_3 - 2(3p^2 + q^2)Y) \tag{88}$$

and

$$Z'' = \frac{1}{2}(-\sqrt{3}q\lambda_8 + 3p\sigma_3) \tag{89}$$

which completes the construction of K. But generally , $Z, Z^{'}, Z^{''}$ are not required to be orthogonal and Randjbar-Daemi et al at 1984 find that we can choose

$$Z = \frac{1}{2}(\sqrt{3}p\lambda_8 + q\sigma_3 + 2rY) \tag{90}$$

but take two U(1) factors of H to be

$$X^{'} = \frac{1}{2}\sqrt{3}\lambda_8 + sY \tag{91}$$

and

$$X^{''} = \frac{1}{2}\sigma_3 + tY \tag{92}$$

where s and t are free parameters,but there is a constraint

$$ps + qt - r \neq 0 \tag{93}$$

which the space is labelled as M^{pqrt}.This space will gives a effective theory which looks like our standard model, but only has free particles.

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